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The unusual electronic spectrum of an infinite quasiperiodic chain: extended signature of all eigenstates

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Abstract. We have shown in an analytical way supported by intuitive arguments that all the eigenstates of an infinite quasiperiodic chain can be of *extended* nature in an unusual way. The well known copper mean chain provides such an example. Earlier works, based on calculations with systems having finite sizes, showed the co-existence of localized and extended states for this particular system. Within the framework of the real space renormalization group scheme we transform a perfectly ordered chain and a copper mean chain into a period doubling sequence for which an area-preserving dynamical map is already in existence. The recursion relations for the Hamiltonian parameters of the effective period doubling sequences generated from an ordered chain and a copper mean chain are then compared to extract information regarding the true nature of the eigenstates of the latter.

1. Introduction

The interest in the nature of electronic eigenstates in one-dimensional (1D) systems without any translational invariance has recently been rekindled, particularly since the success in fabricating aperiodic superlattice structures in the laboratory [1]. While it is well known that for a perfectly ordered chain all the wavefunctions are extended (Bloch functions), whereas for a random distribution of potentials they are exponentially localized, a new class of wavefunctions is found to exist in the case of 1D quasiperiodic sequences [2].

The Fibonacci chain quasicrystal is the most extensively studied example of a 1D quasiperiodic sequence in which the wavefunctions have been categorized as 'critical' (neither localized, nor extended in the usual sense), and the corresponding energy spectrum has been shown to be a Cantor set [2]. However, several extensions and generalizations of the Fibonacci family have attracted considerable attention recently [3] due to variations exhibited in their dynamical behaviour and the excitation spectra. In a particular type of generalization [4] binary quasicrystalline chains are generated by two kinds of inflation rule, viz. $L \rightarrow L^nS, S \rightarrow L$ (class I) and $L \rightarrow LS^n, S \rightarrow L$ (class II) with $n > 1$. L and S are two fundamental building blocks constituting the sequence. $n = 1$ generates the popular golden mean Fibonacci lattice. It has been illustrated [4] that systems belonging to class I exhibit a dynamical area-preserving map similar to the Fibonacci sequence with the golden mean, and the energy spectrum remains, in general, a Cantor set. On the other hand, the dynamical map for systems belonging to class II has been shown to be an area (volume) non-preserving map. Such systems apparently have no constant of motion and their excitation spectra are found to be qualitatively different from those belonging to class

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I. For example, Kolar and Ali [5] have studied magnetic excitations in a copper mean superlattice ($n = 2$ in class II) using finite-sized chains. Their analysis shows that the spectrum of the copper mean superlattice is not Cantor like in the entire frequency regime. They report that the spectrum contains almost continuous clusters of bands with a mixing of self-similar and regular patterns. The electronic spectrum of a simple copper mean chain (CMC) was also shown to have a similar behaviour [6]. Severin *et al* [7] have reported a class of aperiodic systems in 1D in which extended eigenfunctions of both quasiperiodic and periodic nature in a sub-class of the CMC have been found. All these studies were made on finite-sized systems. Chakrabarti and Karmakar [8] dealt with an infinite CMC using real space renormalization group (RSRG) techniques. The infinite CMC can be generated by repeated application of the inflation rule L $\rightarrow$ LSS and S $\rightarrow$ L as mentioned earlier, where L and S stand for two types of bond, say 'long' and 'short'. The lattice thus consists of four different vertices α, β, γ and δ flanked by L–L, L–S, S–L and S–S bonds on both sides [8]. The nearest-neighbour hopping integrals are assigned two values viz. t_L and t_S for hopping across L or S bonds respectively. In this model, the existence of extended eigenstates had been reported [8]. Later it was pointed out by Zhong *et al* [9] that the on-site model of a CMC consisting of two types of atom A and B arranged in the copper mean sequence supports an extended eigenstate at $E = \epsilon_B$, ϵ_B being the on-site potential for the B type atom. In a very recent publication [10] it has been illustrated that the α vertices, always occurring in pairs (α–α) at all length scales of the CMC, give rise to a full hierarchy of extended states each occurring at $E = \epsilon_\alpha^{(n)}$, where the superscript (n) stands for the nth stage of renormalization. The number of allowed energy values increases with the progress of iteration. The totality of all these energies constitutes the entire spectrum of the extended states for the CMC. These eigenenergies form minibands in energy space, the existence of which was detected previously through numerical calculations [8]. Looking carefully at these results we face an extremely interesting situation. The RSRG transformation can, in principle, be carried out to infinite iterations. Each iteration unveils a set of extended states in the energy spectrum of the CMC. The obvious question that arises out of this observation is *does this method exhaust all the eigenstates of the system?* If the answer is 'yes', then the entire eigenvalue spectrum of an infinite CMC will consist of *extended states only*! If the answer is 'no', then can we say anything about the nature of 'other' eigenstates, which may not be a solution of the equation $E = \epsilon_\alpha^{(n)}$?

As is obvious from the above results and the discussion, the true character of the energy spectrum, particularly the nature of the eigenfunctions of a CMC (and consequently, its followers in class II) is far from clearly understood. The scenario appears to be changing in a much more exciting way, particularly since the analysis of Sil *et al* [10]. In this article, we try to look deeper into this aspect of the problem. Our analysis leads to the understanding that, an infinite CMC should sustain *extended electronic eigenstates only*. This, to our mind, is highly exciting because the CMC provides a remarkable example of a 1D chain where, in spite of the absence of any translational invariance, all the eigenstates appear to be delocalized (in a generalized Bloch sense).

A good way of examining the nature of electronic eigenfunctions of a 1D chain has been the RSRG method. The method can be easily implemented in the case of a self-similar lattice such as a CMC. Within the framework of the RSRG scheme one generally observes the flow pattern of the hopping integrals of the Hamiltonian describing the system under successive renormalization. As described in [10], the behaviour of the on-site potential under renormalization also plays an important role in understanding the extended nature of eigenfunctions in systems without translational invariance. However, to check whether a specific energy eigenvalue corresponds to an extended state or not, by studying of the

flow of the Hamiltonian parameters under RSRG, a precise evaluation of the eigenvalue is required. In a quasiperiodic system locating an energy eigenvalue exactly becomes a formidable task, and, becomes practically impossible when the chain length goes to infinity. This is mainly because of the highly fragmented band structure generally encountered in such systems. Any numerically determined eigenvalue for a given finite system size may as well be found to be in a gap (or in the immediate neighbourhood of a gap) for a system with a bigger chain length. The entire flow pattern of the Hamiltonian parameters can therefore change drastically, and can even give wrong information regarding the true nature of the eigenstates unless the true energy eigenvalue is considered.

In the present article we avoid the task of a precise determination of the entire eigenvalue spectrum of a quasicrystalline chain by observing the related dynamical map and the flow of the Hamiltonian parameters under renormalization. In what follows we describe how a known non-linear map of a period doubling sequence can be exploited to achieve this goal.

2. Method and results

We start by considering an infinite period doubling sequence consisting of two types of bond L and S generated according to the inflation rule L $\rightarrow$ LS and S $\rightarrow$ LL. The trace map and the spectral properties of this sequence have already received some attention [11]. The PD sequence can be thought to be a member of the copper mean class in the sense that starting from the initial building block 'L', say, both the lattices can be recursively generated following the rule $G_k = G_{k-1}G_{k-2}G_{k-2}$ for $k \geqslant 2$, with $G_0 = \mathrm{L}$, $G_1 = \mathrm{LS}$ for the PD sequence, and $G_0 = \mathrm{L}$ and $G_1 = \mathrm{LSS}$ for the CMC [8]. G_k stands for the kth generation of the chain. The ratio of the number of L bonds to that of S bonds in the thermodynamic limit is unity for the CMC, whereas it is two for the PD chain. The trace maps for these two systems are of course different [5, 11]. The CMC does not have a polynomial invariant independent of energy, whereas the PD chain has indeed been shown to possess one. However, what has been unobserved so far is that a CMC with two different bonds L and S is a simple 'superlattice' version of a PD chain. One can 'coalesce' two consecutive S bonds in a CMC to get an effective PD chain where the on-site potentials and the hopping integrals become functions of energy in general. To see what this means, one can easily verify that in any arbitrary generation of a CMC the 'S' bonds always come in pairs and the 'L' bonds are either single, or in triplets. On the other hand, in a PD chain, the 'S' bonds are always single, whereas the L come in singlets or triplets only, as in the case of a CMC. If one numbers the bonds from the left, say, in a CMC considering the pairwise occurring S bonds as a single unit, one finds that the L bond and the 'block' of S bonds taken together are indexed exactly in the same way as in a PD chain with S bonds appearing singly. Considering an S–S pair as a single 'unit' in an infinite CMC also brings down the ratio of the number of L and 'SS' blocks to two, the same as that for a PD chain with an isolated S. We illustrate the indexing below:

CMC	L	$\underbrace{\mathrm{SS}}$	L	L	L	$\underbrace{\mathrm{SS}}$	L	$\underbrace{\mathrm{SS}}$	L	$\underbrace{\mathrm{SS}}$	L	L	L	$\underbrace{\mathrm{SS}}$	L	L...
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
PD	L	S	L	L	L	S	L	S	L	S	L	L	L	S	L	L...
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16.

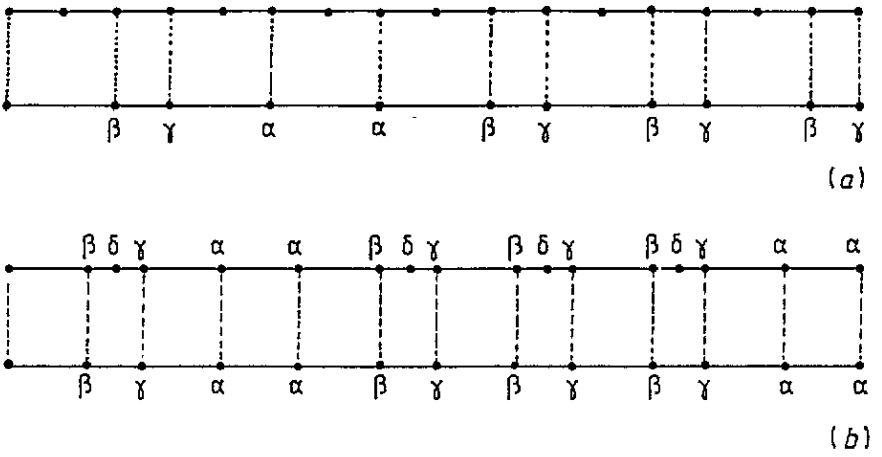

Figure 1. Portions of the infinite chains illustrating the transformations of (*a*) a perfectly ordered chain and (*b*) a copper mean chain into a period doubling chain (the lower line in both cases).

It can be checked that this is true for arbitrarily long chains. Coalescing can be achieved by decimating the δ sites in the CMC, as will be explained later.

A perfectly ordered chain can also be trivially converted into an effective PD chain by forcing the PD sequence on it. We illustrate such transformations in figure 1. Using the deflation rules for the PD chain one can now obtain, within the framework of RSRG techniques, recursion relations for the Hamiltonian parameters of the effective PD chains, which are derived from a CMC and an ordered lattice. Since the basic functional forms of the recursion relations turn out to be the same in both cases, it is really tempting to make a detailed comparative study of the evolution of the parameter space in these two different cases to compare the qualitative nature of the spectrum of the CMC, particularly the nature of the wavefunctions, with that of the well known ordered case. Such a comparative study has already been useful in identifying extended states in a generalization of a golden mean Fibonacci chain [12] and a CMC [10]. We shall proceed along this line. However, before making this comparison we describe some results of the RSRG analysis of the PD chain in a little detail for a better understanding of our arguments.

We describe an infinite PD chain using the standard tight-binding Hamiltonian

$$H = \sum_n \epsilon_n |n\rangle\langle n| + \sum_{(nm)} t_{nm} |n\rangle\langle m| \tag{1}$$

where ϵ_n is the on-site potential at the nth site, and t_{nm} is the nearest-neighbour hopping integral. In the case of a PD sequence the on-site term takes on three values, ϵ_α, ϵ_β and ϵ_γ corresponding to the vertices flanked by L–L, L–S and S–L pairs of bonds (figure 1). The nearest-neighbour hopping integrals can take on two values t_L and t_S corresponding to the hopping of an electron across 'L' and 'S' bonds respectively. We can now use the inflation rule for the PD sequence in the opposite way to decimate a chosen set of sites, which is a standard way of using RSRG techniques [10, 12]. This produces a new chain with the Hamiltonian parameters satisfying a set of recursion relations, viz.

$$\begin{aligned}
\epsilon'_\alpha &= \epsilon_\gamma + t_L^2/(E-\epsilon_\beta) + t_S^2/(E-\epsilon_\beta) \\
\epsilon'_\beta &= \epsilon_\gamma + t_L^2/(E-\epsilon_\alpha) + t_S^2/(E-\epsilon_\beta) \\
\epsilon'_\gamma &= \epsilon_\alpha + t_L^2/(E-\epsilon_\alpha) + t_L^2/(E-\epsilon_\beta) \\
t'_L &= t_L t_S/(E-\epsilon_\beta) \\
t'_S &= t_L^2/(E-\epsilon_\alpha).
\end{aligned} \tag{2}$$

We can now reduce the dimensionality of the parameter space by defining

$$W = (E-\epsilon_\alpha)/t_L$$

$$X = (E-\epsilon_\beta)/t_L$$

$$Y = (E-\epsilon_\gamma)/t_L$$

and

$$Z = t_S/t_L.$$

W, X, Y and Z are then found to satisfy the following non-linear recursion relations:

$$\begin{aligned}
W_{n+1} &= X_n Y_n/Z_n - 1/Z_n - Z_n \\
X_{n+1} &= X_n Y_n/Z_n - Z_n - X_n/W_n Z_n \\
Y_{n+1} &= W_n X_n/Z_n - 1/Z_n - X_n/W_n Z_n \\
Z_{n+1} &= X_n/W_n Z_n.
\end{aligned} \tag{3}$$

Here, the subscript n refers to the stage of the RSRG operation. Using the set of equations (3) we have been able to calculate a quantity

$$I_{PD} = W_{n+2}/2 - (W_n^2 - 2)W_{n+1}/2 + 1$$

which is independent of the iteration index n, i.e. remains invariant under successive RSRG operations. I_{PD} is also found to be independent of energy E in this case. The existence of such an invariant generally indicates a Cantor set energy spectrum and critical wavefunctions, as has been observed earlier in the case of a Fibonacci chain with a different approach [2]. In our case I_{PD} plays an important role as will be clear from the discussion that follows.

Our objective in the present investigation, however, is simply to use the non-linear recursion relations (3) together with the invariant to unravel the nature of eigenfunctions of the CMC, so we leave aside the detailed study of the map (3) as a separate issue. Instead, we first take a perfectly ordered chain with site energy ϵ and nearest-neighbour hopping integral t and force the PD sequence onto it (figure 1(*a*)), as proposed earlier, to map it 'artificially' onto an effective PD chain with site energies and hopping integrals given by

$$\begin{aligned}
\epsilon_\alpha^{(0)} &= \epsilon + 2t^2/(E-\epsilon) \\
\epsilon_\beta^{(0)} &= \epsilon_\gamma^{(0)} = \epsilon + t^2/(E-\epsilon) \\
t_L^{(0)} &= t^2/(E-\epsilon) \\
t_S^{(0)} &= t.
\end{aligned} \tag{4}$$

The superscript (0) now stands for the initial values of the Hamiltonian parameters for the effective PD chain. *It is important to appreciate that, since we have forced the ordered chain to have an effective PD chain configuration, the evolution of the entire parameter space under RSRG according to equations (2) and (3), with the initial values given by equation (4), will give us information about the nature of eigenstates of the ordered chain only.* Within the same framework one can easily calculate the density of states (DOS) to check that the DOS for the ordered chain is exactly reproduced as expected. It is a well established fact that all the eigenstatets of a perfectly ordered linear chain are extended (Bloch functions). When we study the flow pattern of the Hamiltonian parameters under RSRG iterations for such an effective PD chain, we observe that for any energy E within the band, at every stage n of renormalization, we have the relationship $\epsilon_\alpha^{(n)} \neq \epsilon_\beta^{(n)} = \epsilon_\gamma^{(n)}$, and that the hopping integals never flow to zero, indicating that at any scale of length there is a non-vanishing connection between nearest neighbours (on that scale of length), i.e. states are all extended [12]. Such a typical pattern exhibited by the renormalized values of the site energies and the hopping integrals can therefore be taken to be a signature of extended eigenfunctions [12, 13]. A similar observation can be made in the case of a Fibonacci chain with the golden mean. Taking the ordered limit, i.e. setting $\epsilon_i = 0$ for $i = \alpha, \beta, \gamma$ and $t_L = t_S$, one finds that the parameter space evolves in the same way. However, taking the ordered limit of a quasiperiodic chain may prove to be different from generating an 'effective' quasiperiodic chain by forcing the appropriate inflation rule on a periodic lattice. For example, the ordered limit of the PD sequence yields $I_{PD} = 0$ independent of energy, whereas the invariant in the 'effective' PD in our case turns out to be energy dependent. The invariant corresponding to this effective PD chain can be easily obtained by selecting the initial values for the on-site terms and the hopping integrals from (4), and the initial value (and subsequent values also) of the invariant is given by

$$I_{ORD} = (-\tfrac{1}{2}t)E + (1 + \epsilon/2t).$$

I_{ORD} is a continuous function of energy, but remains finite and fixed at every iteration for E lying within the band only.

We now turn our attention to the CMC. As has already been pointed out, we find that in a CMC defined by two types of bond L and S (with corresponding hopping integrals t_L and t_S respectively) we can classify the site energies as ϵ_α, ϵ_β, ϵ_γ and ϵ_δ [8]. If we now decimate the δ type sites, we end up with an 'effective' PD chain with the site energies and the hopping integrals given by

$$\begin{aligned}
\epsilon_\alpha^{(0)} &= \epsilon_\alpha \\
\epsilon_\beta^{(0)} &= \epsilon_\beta + t_S^2/(E - \epsilon_\delta) \\
\epsilon_\gamma^{(0)} &= \epsilon_\gamma + t_S^2/(E - \epsilon_\delta) \\
t_L^{(0)} &= t_L \\
t_S^{(0)} &= t_S^2/(E - \epsilon_\delta).
\end{aligned} \tag{5}$$

Recalling our earlier discussion in the context of an ordered chain, we now know that the evolution of the parameter space of this 'new' PD chain will now give us all the information about the nature of eigenstates of a CMC. Once again, we have verified that the CMC density of states is exactly reproduced, as previously obtained [8]. This is of course expected since the underlying lattice is basically a CMC. One can now look at the nature of the invariant

associated with the non-linear mapping for this new PD chain generated from a CMC. It is extremely interesting to observe that a straightforward algebra yields

$$I_{\mathrm{CMC}} = mE + N$$

where

$$m = (\epsilon_\beta\epsilon_\gamma - \epsilon_\alpha\epsilon_\beta - \epsilon_\alpha\epsilon_\gamma + \epsilon_\alpha^2 - t_{\mathrm{L}}^2)/2t_{\mathrm{L}}t_{\mathrm{S}}^2$$

and

$$N = \{(\epsilon_\beta + \epsilon_\gamma - 2\epsilon_\alpha + 2t_{\mathrm{L}})/2t_{\mathrm{L}} - \epsilon_\delta(\epsilon_\beta\epsilon_\gamma - \epsilon_\alpha\epsilon_\beta - \epsilon_\alpha\epsilon_\gamma + \epsilon_\alpha^2 - t_{\mathrm{L}}^2)/2t_{\mathrm{L}}t_{\mathrm{S}}^2\}.$$

In this expression for I_{CMC} the on-site terms and the hopping integrals refer to those in the original CMC. The striking equivalence in the forms of the invariants I_{ORD} and I_{CMC} can now be exploited to extract information about the eigenstates of the latter chain as explained in the following section.

3. Discussion

One can always tune the intial values of the on-site terms and the hopping integrals for the CMC so as to obtain an identical value for the invariant as that in the case of a perfectly ordered chain with some other values of ϵ and t. The invariant remains unchanged, and finite at every stage of renormalization for all energy eigenvalues. This simple statement, in our view, has a deeper significance. An identical invariant for the ordered chain and the CMC in this situation means that the parameter space for these two different cases must have evolved in the same qualitative manner (of course, not with the same values for different parameters) so that the tip of the four-dimensional vector with 'components' W, X, Y and Z remains on the same invariant 'line' for both the lattices at each stage of renormalization. It is highly important to appreciate that in this scheme we are simply using the recursion relations for the PD chain as a 'black box', and if the input values for the Hamiltonian parameters from two differnt systems going into this 'black box' yield identical invariant 'surfaces' in the parameter space, it will be impossible to distinguish, from a renormalization group point of view, between the qualitative nature of the eigenfunctions of those two systems. The above idea can be clarified even more taking specific models.

Table 1. List of a few eigenvalues of a CMC with $\epsilon_i = 0$, $i = \alpha, \beta, \gamma, \delta$ and $t_{\mathrm{S}}/t_{\mathrm{L}} = 2$ for which there always exists a corresponding eigenvalue of a perfectly ordered chain with $\epsilon = 0$ and $t = -1$. The invariants for the two cases are found to be exactly equal to each other.

Some of the eigenvalues of a CMC	Corresponding eigenvalues of an ordered chain	Invariant in both cases
0.0	0.0	1.0
-13.6379×10^{-1}	-34.0950×10^{-2}	0.8295
13.6379×10^{-1}	34.0950×10^{-2}	1.1704
-3.0	-74.9988×10^{-2}	0.6250
3.0	75.0979×10^{-2}	1.3755
-30.2325×10^{-1}	-75.5820×10^{-2}	0.6221
30.2325×10^{-1}	75.5810×10^{-2}	1.3779

3.1. The transfer model

We argue for a transfer model version of the CMC. Setting all the ϵ values to zero and $t_S/t_L = 2$, we find that

$$I_{CMC} = (-\tfrac{1}{8}t_L)E + 1.$$

The form is exactly equivalent to that of an ordered chain. Not only that, but even looking at the initial values of the Hamiltonian parameters for the PD chain obtained from the CMC, we find that $\epsilon_\alpha^{(0)} \neq \epsilon_\beta^{(0)} = \epsilon_\gamma^{(0)}$, a pattern typically observed at the initial stage, and subsequently at all stages of renormalization in the case when a PD chain is created from a perfectly ordered chain. Therefore, for some initial ϵ and t the evolutions of the entire parameter space in these two basically different transformations (i.e. ordered $\rightarrow$ PD and CMC $\rightarrow$ PD) start with the same pattern in the distribution of the on-site potentials and hopping terms, and can end up with the identical invariant. *This happens for every energy eigenvalue for the CMC, as for any value of such energy eigenvalues one can always define ϵ and t for a corresponding ordered chain to achieve $I_{CMC} = I_{ORD}$.* for example, we have provided in table 1 a list of several selected eigenvalues for the CMC together with the value of I_{CMC} in each case. In the same table we also exhibit those eigenenergies of a perfectly ordered chain *with $\epsilon = 0$ and $t = -1$ that give rise to an identical invariant for the corresponding* CMC case. Though the CMC eigenvalues illustrated in the table are all solutions of the equation $E = \epsilon_\alpha^{(n)}$, and therefore are naturally extended, the mapping can be done for all eigenvalues. It means that the 'black box' containing the recursion relations (2) (or, equivalently, (3)), if fed with similar distributions in the values of the Hamiltonian parameters in the two cases, becomes insensitive to the nature of the distribution in the renormalized values of the parameters, and consequently to the nature of the eigenfunctions. This strongly suggests that *the eigenfunctions for the CMC, therefore, cannot be distinguished from those of an ordered lattice as far as their extended nature is concerned.* By the word 'extended' we of course do not mean Bloch functions, which are only possible in translationally invariant systems. Such states can be termed 'generalized Bloch functions'. It is however important to realize that in the case when the parent lattice is an ordered one, the desired flow pattern sets in from the very first stage of renormalization. In the case of a CMC, the relationship beween the site energies sets in like that of an ordered lattice only at a particular length scale and then continues to do so. The length scale at which this similarity is observed for the first time depends on the energy eigenvalue. For example, selecting the parameters for the ordered lattice as $\epsilon = 0$ and $t = -1$ we find that for $E = 0$, $I_{CMC} = I_{ORD}$ from the very beginning of the RSRG operation and the flow pattern in the parameters for the effective period doubling chain derived from a CMC starts giving the desired indication of an ordered like behaviour (i.e. $\epsilon_\alpha^{(n)} \neq \epsilon_\beta^{(n)} = \epsilon_\gamma^{(n)}$ and $t_L = t_S$) from the very first stage, i.e. from $n = 1$ for the effective PD chain. On the other hand, the equality (inequality) in the values of ϵ (t)is observed from the third step of RSRG transformation for $E = 3$ and from step 4 for $E = 1.363\,798\,025\,8$, and so on, when the input comes from the original CMC. This observation actually indicates a genealogical difference between various extended eigenfunctions in a CMC arising out of the α–α clustering at different length scales, as pointed out in [10]. What is interesting from the present point of view is that all the finite values of I_{CMC} should lie between zero and two, as can be checked observing the global band edges of a CMC electronic density of states spectrum [8], and that each point in the I_{CMC}–E line for the effective PD chain (derived from the CMC) then lies in the neighbourhood of a corresponding point in the I_{ORD}–E line (of the PD obtained from an ordered lattice). The point for the ordered lattice acts as an 'attractor' and as the scaling goes on the point corresponding to any CMC eigenvalue is eventually

attracted and ultimately collapses onto a specific point on the invariant line for the ordered chain. If we study the flow in the parameter space only, we need to know the precise value of the energy eigenvalue being investigated. Quasiperiodic systems in general exhibit highly fragmented band structure, and any direct diagonalization of finite-size matrices may yield eigenvalues that fall in the vicinity of a gap in the spectrum for the infinite chain as mentioned earlier. The invariant in this case blows up under iteration and the desired flow pattern cannot be obtained. A simultaneous study of the behaviour of the invariant together with the flow of ϵ and t values does not necessitate an exact determination of the eigenvalues. We thus emphasize the fact that the basic qualitative nature of all the CMC eigenstates should be the same as that for an ordered chain.

3.2. The on-site model

This model is defined by making $\epsilon_\alpha = \epsilon_\gamma$ and $\epsilon_\beta = \epsilon_\delta$, with $t_L = t_S$. Now it becomes interesting to perform a little easy algebra to see that the typical pattern $\epsilon_\alpha^{(n)} \neq \epsilon_\beta^{(n)} = \epsilon_\gamma^{(n)}$ signalling the onset of an 'ordered-like' behaviour is established in the case of the resulting effective PD chain not at $n = 0$, but at $n = 1$. That is, the 'input' into the recursion relations (3) for a scaled version of the effective PD chain becomes the same as if they were obtained from an ordered chain. However, once the pattern is set, we can have an identical invariant for the two cases by suitably choosing the values of ϵ and t in the two cases (CMC and ordered chain), and our previous argument follows in an identical fashion.

Two comments in this regard are in order.

(1) It should be appreciated that I_{ORD} is not obtained by simply putting the ordered limit in the invariant for the true PD sequence. Rather, I_{ORD} is obtained by transforming an ordered lattice into a PD lattice, thereby changing only the *appearance* of the ordered lattice while keeping the exact nature of the energy spectrum and the eigenfunctions intact. The evolution of the parameter space for this transformed lattice, though using the set of recursion relations of a PD chain, thus brings out the exact nature of the wavefunctions of an ordered chain. So, to our mind, identifying I_{CMC} with this I_{ORD} is a meaningful idea to identify the true character of the eigenstates.

(2) If we scan all possible energies for a CMC, we can hit upon energy values that are either in a gap, or, an eigenvalue. For E in a gap I_{CMC} diverges as the iteration proceeds, and therefore is not at all an 'invariant'. For the eigenvalues with a non-zero value of the density of states, since one can always figure out a truly ordered chain in the 'background' that for some combination of ϵ and t gives rise to the same invariant, and hence qualitatively the same flow pattern in the parameter space, the CMC eigenfunctions can only be of *extended* type. The possibility of any *localized/critical wavefunctions* is then excluded.

In conclusion, we have drawn attention to the remarkable fact that, in spite of having no translational invariance, a quasiperiodic CMC should have all its eigenfunctions extended. We have proceeded in a novel analytical way followed by intuitive arguments. Other generalizations in the copper mean family are also strong candidates for having similar spectral properties. Work in this direction is in progress.

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